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Simultaneously Continuous Retractions on the Unit Ball of a Banach Space

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INTRODUCTION

Let X be a separable Banach space, and let $B(X^*)$ be the closed unit ball of its dual X^*. We shall consider X^* as simultaneously equipped with the norm and ω^* topologies, and study the following general problem: *For which spaces X does there exist a retraction from X^* onto $B(X^*)$ which is simultaneously continuous with respect to the norm and ω^* topologies?*

When each of the topologies is considered separately, the existence of continuous retractions is easy to prove and well known. The two topologies problem is much more delicate. It turns out that for some spaces there exists such a retraction, while for others there is none. Moreover, in some spaces one can construct an ω^* continuous retraction which is uniformly continuous with respect to the norm topology. In some spaces we can even compute the "best possible" norm-modulus of continuity of an ω^* continuous retraction. On the other hand, there are spaces for which a simultaneously continuous retraction exists, but there is no ω^* continuous retraction which is norm-uniformly continuous.

Before we go on and describe the contents of the article, we wish to comment on the origin of the problem and its relation to approximation theory. In a recent article [1] (see also Section 4), we needed, for $X = C(K)$, K a compact metric space, an ω^* continuous retraction $\phi: X^* \to B(X^*)$ with the following additional property: For each $x^* \in X^*$, *$\phi(x^*)$ is a nearest point to x^* in $B(X^*)$.*

This is already a result of a similar nature to the problems studied here. We must consider X^* with its two topologies, ϕ is required to be ω^* continuous and to satisfy some condition with respect to the norm topology.

The nearest point map considered above is very special for $X = C(K)$, and

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cannot be constructed in most spaces. Indeed, assume for example that X^* is strictly convex; then $x^*/\|x^*\|$ is the *unique* nearest point in $B(X^*)$ for $x^* \notin B(X^*)$. But the retraction $x^* \to x^*/\|x^*\|$ (for $x^* \notin B(X^*)$) is not ω^* continuous for any infinite-dimensional X! (see Section 1).

We are thus led to consider approximation nearest points. Let $f(t)$ be a nonnegative function, defined for $t > 0$, such that $f(t) \to 0$ when $t \to 0$.

DEFINITION. A map $\phi: X^* \to B(X^*)$ is called an *f-approximate nearest point map* if $\|\phi(x^*) - x^*\| \leqslant d(x^*, B(X^*)) + f(d(x^*, B(X^*)))$ for all $x^* \in X^*$.

Since we require that $f(t) \to 0$, this is a very strong notion of approximate nearest point. Using this notion we can formulate a meaningful question about ω^* continuous approximate nearest point maps: *Given a separable Banach space X, can one find an f, and an ω^* continuous f-approximate nearest point map from X^* onto $B(X^*)$?* If so, an interesting problem is to find a "best possible" f.

This question is strongly related to the problem of finding an ω^* continuous retraction which is norm-uniformly continuous. Indeed, if ϕ is such a retraction, with norm-modulus of continuity ω_ϕ, then a simple computation (see Lemma 1.2) shows that ϕ is an ω_ϕ-approximate nearest point map. Thus all our results giving estimates for the possible norm-modulus of continuity of ω^* continuous retractions can be viewed as results on the possible degree of ω^* continuous approximation of nearest points.

We now describe the content of the various sections. After some preliminaries in Section 1, we construct in Section 2 a simultaneously continuous retraction in X^* when X has a shrinking basis whose dual basis is strictly monotone. This is used to construct uniformly simultaneously continuous retractions on l_p and to estimate their norm-modulus of continuity.

The main result in Section 3 is that if X^* is uniformly convex with modulus of convexity $\delta(\varepsilon)$, then there is an ω^* continuous retraction from X^* onto $B(X^*)$ which is norm-uniformly continuous with norm-modulus of continuity $\delta^{-1}(t)$. This is used to estimate the norm-modulus of continuity of an ω^*-continuous retraction on L^p.

In Section 4 we prove that for K compact metric, $C(K)^*$ admits an ω^*-continuous retraction ϕ satisfying $\|\phi(\mu) - \phi(\nu)\| \leqslant 2\,\|\mu - \nu\|$ for all $\mu, \nu \in C(K)^*$.

In the final section, we give lower estimates for the possible norm-modulus of continuity of an ω^* continuous retraction in some spaces. These allow us to show that the estimates obtained for l_p and L_p are the best possible, and also to construct various counterexamples.

Our notation is standard, see, e.g., [3, 4]. We only treat the real case. The modifications for complex scalars are straightforward.

1. Preliminaries

Let X be a Banach space. There is a natural retraction r from X^* onto $B(X^*)$:

$$
\begin{aligned}
r(x^*) &= x^*, & x^* &\in B(X^*), \\
&= x^*/\|x^*\|, & x^* &\notin B(X^*).
\end{aligned}
$$

It is easy to check that r is norm-continuous. In fact, it satisfies a Lipschitz condition with constant at most 2.

To appreciate the problem dealt with in this paper, it is important to realize that whenever X is infinite-dimensional, r is *not* ω^* continuous. Indeed, when X is infinite-dimensional, the sphere $S(X^*) = \{x^* : \|x^*\| = 1\}$ is ω^* dense in $B(X^*)$. So fix any x^*, $\|x^*\| = \frac{1}{2}$, and choose a net $x_\alpha^* \in S(X^*)$ so that $x_\alpha^* \to^{\omega^*} x^*$. Then $r(2x_\alpha^*) = x_\alpha^* \to^{\omega^*} x^* \neq 2x^* = r(2x^*)$, although $2x_\alpha^* \to^{\omega^*} 2x^*$.

Thus the construction of an ω^* continuous retraction requires a more subtle approach. The standard proof that for separable X, an ω^* continuous retraction from X^* onto $B(X^*)$ exists uses Michael's selection theorem [4]. Moreover, when X is nonseparable, there are cases where there is no ω^* continuous retraction from X^* onto $B(X^*)$. (See, e.g., [2], where a somewhat stronger result is proved for nonseparable Hilbert space.)

General selection theorems are not suitable for the two-topologies problems we deal with, and our approach will be more elementary. Using the structure or geometry of the spaces involved, we construct the retractions directly.

We introduce the following terminology: We shall say that X^* admits a simultaneously continuous retraction if there is a simultaneously norm and ω^* continuous retraction ϕ from X^* onto $B(X^*)$. If the ω^* continuous retraction ϕ is uniformly continuous with respect to the norm topology, with norm-modulus of continuity $\omega_\phi(t) = \sup\{\|\phi(x^*) - \phi(y^*)\| : \|x^* - y^*\| \leqslant t\}$ satisfying $\omega_\phi(t) \leqslant f(t)$, we shall say that X^* admits a uniformly simultaneously continuous retraction or, more specifically, that X^* admits an f-uniformly simultaneously continuous retraction.

We finish this section with two simple lemmas.

Lemma 1.1. *Let X be a norm one complemented subspace of Y. If Y^* admits a (f-uniformly) simultaneously continuous retraction, so does X^*.*

Proof. Let $\phi: Y^* \to B(Y^*)$ be the retraction, and let $P: Y \to X$ be a norm one projection. The desired retraction $\psi: X^* \to B(X^*)$ is defined by $\psi(x^*) = \phi(P^*x^*)|_X$.

LEMMA 1.2. *Let ϕ be an f-uniformly simultaneously continuous retraction onto $B(X^*)$. Then ϕ is an f-approximate nearest point map.*

Proof. Given $x^* \notin B(X^*)$, $\phi(x^*/\|x^*\|) = x^*/\|x^*\|$ because ϕ is a retraction. Also $\big\|x^* - x^*/\|x^*\|\,\big\| = \|x^*\| - 1 = d(x^*, B(X^*))$. Thus

$$\begin{aligned}\|x^* - \phi(x^*)\| &\leqslant \big\| x^* - x^*/\|x^*\|\,\big\| + \big\| \phi(x^*/\|x^*\|) - \phi(x^*)\big\| \\ &\leqslant d(x^*, B(X^*)) + \omega_\phi(d(x^*, B(X^*))) \\ &\leqslant d(x^*, B(X^*)) + f(d(x^*, B(X^*))).\end{aligned}$$

2. Strictly Monotone Schauder Bases

Let $\{e_j\}$ be a normalized Schauder basis for X with biorthogonal functionals $\{e_j^*\}$, and associated projections P_n. Recall that the basis is called monotone if $\|P_n\| = 1$ for all n. It is called strictly monotone if $\|P_n x\| < \|x\|$ whenever $(I - P_n)x \neq 0$. The basis is called shrinking if $\{e_j^*\}$ is a basis for X^*.

THEOREM 2.1. *Let X have a shrinking Schauder basis $\{e_j\}$ with $\{e_j^*\}$ being strictly monotone. Then X^* admits a simultaneously continuous retraction.*

Proof. Given $x^* = \sum a_j e_j^* \in X^*$ with $x^* \notin B(X^*)$, there is by the strict monotonicity, a unique n so that $\|\sum_1^{n-1} a_j e_j^*\| < 1$ and $\|\sum_1^n a_j e_j^*\| \geqslant 1$. By the strict monotonicity again, there is a unique $0 < t \leqslant 1$ so that $\|\sum_1^{n-1} a_j e_j^* + t a_n e_n^*\| = 1$, and we define $\phi(x^*) = \sum_1^{n-1} a_j e_j^* + t a_n e_n^*$. Defining $\phi(x^*) = x^*$ for $x^* \in B(X^*)$, it is easy to check (again by the strict monotonocity) that ϕ is a ω^* continuous retraction on $B(X^*)$.

By the definition of ϕ, $\|(I - P_n^*)\,\phi(x^*)\| \leqslant \|(I - P_n^*)\,x^*\|$ for every x^* and every n. Assume now that $\|x_k^* - x^*\| \to 0$ and fix $\varepsilon > 0$. Choose n so that $\|(I - P_n^*)\,x_k^*\|$, $\|(I - P_n^*)\,x^*\| < \varepsilon/3$ for all k, and then use the ω^* continuity of ϕ to find k_0 so that $\|P_n^*(\phi(x_k^*) - \phi(x^*))\| < \varepsilon/3$ for all $k > k_0$. Thus $\|\phi(x_k^*) - \phi(x^*)\| < \varepsilon$ for all $k > k_0$, proving the norm continuity of ϕ.

Theorem 2.1 gives only simultaneously continuous retraction. In general one cannot obtain uniform simultaneous continuity (see Example 5.4), but for specific spaces the construction may yield precise estimates on the norm modulus of continuity. We do this in the next theorem for l_p.

THEOREM 2.2. *Let $1 < p < \infty$; then l_p admits a uniformly simultaneously continuous retraction ϕ, with norm-modulus of continuity $\omega_\phi(t) \leqslant c t^{1/p}$.*

Proof. Specializing the construction of Theorem 2.1 to l_p we see that if $x=(a_1, a_2,\ldots)\in l_p$ then the nth coordinate of $\phi(x)$ is given by

$$\begin{aligned}
(\phi(x))_n &= a_n, & \sum^n |a_j|^p &\leqslant 1,\\
&= 0, & \sum^{n-1} |a_j|^p &\geqslant 1,\\
&= \left(1-\sum^{n-1}|a_j|^p\right)^{1/p} \operatorname{sign} a_n, & \sum^{n-1} |a_j|^p &\leqslant 1 \leqslant \sum^n |a_j|^p.
\end{aligned}$$

We first prove that for all $x\notin B(l_p)$, $\|\phi(x)-x\|^p\leqslant\|x\|^p-1$.

Indeed write $x=(x_1,\ldots,x_n,\ldots)$ and assume $\phi(x)=(x_1,\ldots,x_{n-1},a,0,0,\ldots)$ (i.e. $\sum_1^{n-1}|x_j|^p<1\leqslant\sum_1^n|x_j|^p$ and $\sum_1^{n-1}|x_j|^p+|a|^p=1$).

Then a and x_n have the same sign and $|a|\leqslant|x_n|$. Thus $|x_n-a|^p\leqslant|x_n|^p-|a|^p=\sum_1^n|x_j|^p-1$, and $\|x-\phi(x)\|^p=|x_n-a|^p+\sum_{n+1}^\infty|x_j|^p\leqslant\sum_1^\infty|x_j|^p-1=\|x\|^p-1$.

Assume now that $x=(x_1,\ldots)$ and $y=(y_1,\ldots)$ satisfy $\|x-y\|\leqslant\varepsilon$ and write $\phi(x)=(x_1,\ldots,x_{n-1},a,0,0,\ldots)$, $\phi(y)=(y_1,\ldots,y_{m-1},b,0,0,\ldots)$. Without loss of generality we can assume that $m\geqslant n$, and then by changing the coordinates of x and y *after* the mth position, we can assume that in fact $x=(x_1,\ldots,x_m,0,0,\ldots)$ and $y=(y_1,\ldots,y_m,0,\ldots)$. Moreover, we can replace y_m by b and x_m by x_m-y_m+b, without changing $\|x-y\|$ and the values $\phi(x)$ and $\phi(y)$. (This is true if $m>n$. If $n=m$ we may have to change the roles of x and y to do this).

Having done all these reductions, we see that we can assume that $\|y\|\leqslant 1$, which implies that $\phi(y)=y$, and then of course $\|x\|\leqslant 1+\varepsilon$. Thus $(\|x\|^p-1)^{1/p}\leqslant C\varepsilon^{1/p}$ and $\|\phi(x)-\phi(y)\|\leqslant\|\phi(x)-x\|+\|x-y\|\leqslant(\|x\|^p-1)^{1/p}+\varepsilon\leqslant c\varepsilon^{1/p}$.

Remarks. (1) We shall see in Section 5 that the theorem gives the best possible estimate on the norm-modulus of continuity of ω^* continuous retraction in l_p.

(2) The same proof shows that for l_1, consider as the dual of c_0, the retraction obtained satisfies a Lipschitz condition: $\|\phi(x)-\phi(y)\|\leqslant 2\|x-y\|$.

(3) The norm-moduli of continuity of l_p get worse as $p\to\infty$, and of course Theorem 2.1 does not apply to l_∞ (as the dual of l_1). But l_∞ admits a Lipschitz one ω^* continuous retraction: Let $f(t)=\operatorname{sign} t\min(|t|,1)$ and define, for $x=(x_1,x_2,\ldots)\in l_\infty$, $\phi(x)=(f(x_1),f(x_2),\ldots)$. It is easy to check that ϕ is ω^* continuous and satisfies $\|\phi(x)-\phi(y)\|\leqslant\|x-y\|$.

3. Uniformly Convex Spaces

A Banach space Y is called uniformly convex if for every $\varepsilon > 0$, one has $\delta(\varepsilon) > 0$, where $\delta(\varepsilon) = \inf\{2 - \|x+y\| : x, y \in B(Y), \ \|x-y\| \geqslant \varepsilon\}$. The function $\delta(\varepsilon)$ is called the *modulus of convexity* of Y.

THEOREM 3.1. *Let X be a separable Banach space such that X^* is uniformly convex with modulus of convexity $\delta(\varepsilon)$. Then X^* admits a δ^{-1}-uniformly simultaneous continuous retraction.*

Proof. Let $E_1 \subset E_2 \subset \cdots$ be a sequence of finite-dimensional subspaces of X such that $\dim E_n = n$ and such that $\bigcup E_n$ is dense in X. We introduce the following notation: $R_n : X^* \to E_n^*$ is the restriction operator, i.e., $R_n x^*$ is the element in E_n^* satisfying $\langle x, R_n x^* \rangle = \langle x, x^* \rangle$ for all $x \in E_n$. Also, $\psi_n : E_n^* \to X^*$ is the Hahn–Banach extension, i.e., for each $x^* \in E_n^*$, $\psi_n(x^*)$ is the unique (since X^* is strictly convex) Hahn–Banach extension of x^* to X^*. We denote by $\tau_n : E_n^* \to E_{n+1}^*$ the Hahn–Banach extension from E_n^* to E_{n+1}^*, i.e., $\tau_n = R_{n+1} \circ \psi_n$. For each $x^* \notin B(X^*)$ put $n(x^*) = \inf\{n : \|R_n x^*\| \geqslant 1\}$.

Let $x^* \notin B(X^*)$ and assume $n(x^*) = n > 1$. Then $R_n x^*$ and $\tau_{n-1}(R_{n-1}x^*)$ are both elements of E_n^* whose restriction to E_{n-1} is $R_{n-1}x^*$, with $\|\tau_{n-1}(R_{n-1}x^*)\| = \|R_{n-1}x^*\| < 1$ and $\|R_n x^*\| \geqslant 1$. Thus there is a unique $0 < \lambda \leqslant 1$ such that if we put $z^* = z^*(x^*) = \lambda R_n x^* + (1-\lambda)\,\tau_{n-1}(R_{n-1}x^*)$, then $\|z^*\| = 1$, and of course the restriction of z^* to E_{n-1} is also $R_{n-1}x^*$.

We are now ready to define the retraction:

$$
\begin{aligned}
\phi(x^*) &= x^*, && x^* \in B(X^*),\\
&= \psi_1(R_1 x^*/\|R_1 x^*\|), && x^* \notin B(X^*) \ \text{ and } \ n(x^*) = 1,\\
&= \psi_n(z^*), && x^* \notin B(X^*) \ \text{ and } \ n(x^*) = n > 1
\end{aligned}
$$

(where $z^* = z^*(x^*)$ is defined as above). Clearly $\phi : X^* \to B(X^*)$ and is the identity on $B(X^*)$, and we first check the norm-modulus of continuity of ϕ.

Fix $x^*, y^* \in X^*$ with $\|x^* - y^*\| = \varepsilon$, and assume that $n = n(x^*) \leqslant n(y^*) = m$. We distinguish several cases.

Case 1, $1 < n < m$. Write $\phi(x^*) = \psi_n(z^*)$, where $z^* = \lambda R_n x^* + (1-\lambda)\,\tau_{n-1}(R_{n-1}x^*)$ is in E_n^*. Since $\|z^*\| = 1$, there is an $x \in E_n$ with $\|x\| = \langle x, z^* \rangle = 1$. Now z^* is a convex combination of $R_n x^*$ with an element whose norm is strictly smaller than 1, thus necessarily $\langle x, R_n x^* \rangle \geqslant 1$, and so $\langle x, R_n y^* \rangle \geqslant \langle x, R_n x^* \rangle - \|R_n(x^* - y^*)\| \geqslant 1 - \varepsilon$. By the definition of ϕ, $\phi(y^*)$ is an extension of $R_n y^*$, and this gives that $\|\phi(x^*) + \phi(y^*)\| \geqslant \langle x, R_n(\phi(x^*) + \phi(y^*)) \rangle = \langle x, z^* + R_n y^* \rangle \geqslant 2 - \varepsilon$.

By the definition of δ, this implies that $\|\phi(x^*)-\phi(y^*)\| \leqslant \delta^{-1}(\varepsilon)$.

Before passing to the next case, we notice that the fact that $m>n$ was used only to ensure that $\phi(y^*)$ is an extension of $R_n y^*$.

Case 2, $\infty > m = n > 1$. Since E_n contains E_{n-1} as a subspace of co-dimension one, there is a $u^* \in X^*$ which annihilates E_{n-1}, and such that every element in E_n^* which annihilates E_{n-1} is a multiple of $v^* = R_n u^*$. We shall now replace x^* and y^* by $x_1^* = x^* + \alpha u^*$, $y_1^* = y^* + \alpha u^*$ for an appropriately chosen α.

Onviously $\|x_1^* - y_1^*\| = \|x^* - y^*\| = \varepsilon$, and also $R_{n-1}x^* = R_{n-1}x_1^*$, $R_{n-1}y^* = R_{n-1}y^*$ (because $R_{n-1}u^* = 0$).

Since $R_n y^*$ and $\tau_{n-1}(R_{n-1}y^*)$ have the same restriction $(R_{n-1}y^*)$ to E_{n-1}, their difference is a multiple of v^*, and there is an α s.t. $\|R_n(y^* + \alpha u^*)\| = \|R_n(y_1^*)\| = 1$. By interchanging the role of x^* and y^* if necessary, we can assume that for this α, $\|R_n x_1^*\| = \|R_n(x^* + \alpha u^*)\| \geqslant 1$.

But now the argument of Case 1 applies to the new x_1^* and y_1^*. Indeed, by the above construction $n(x_1^*) = n(y_1{}^*) = n$ and $\|R_n y_1^*\| = 1$. Thus $R_n\phi(y_1^*) = R_n y_1^*$ and by the remark at the end of Case 1, this is all that is needed to prove that $\|\phi(x_1^*) - \phi(y_1^*)\| \leqslant \delta^{-1}(\varepsilon)$. Since $\phi(x_1^*) = \phi(x^*)$ and $\phi(y_1^*) = \phi(y^*)$, the result follows.

Case 3, $n = m = \infty$. This case is trivial, because $\phi(x^*) = x^*$; $\phi(y^*) = y^*$.

Case 4, $n = m = 1$. In this case either $\phi(x^*) = \phi(y^*)$ or $\phi(x^*) = -\phi(y^*)$, and the second case is possible only when $\|x^* - y^*\| \geqslant 2$, making the estimate trivial.

Case 5, $m > n = 1$. The proof of Case 1 needs only small modifications when $n = 1$. Here we use the fact that $\phi(x^*) = \phi(z^*)$, where $z^* = R_1 x^* / \|R_1 x^*\|$ is a convex combination of $R_1 x^*$ and zero.

We now prove that ϕ is ω^* continuous. Assume that $x_\alpha^* \to^{\omega^*} x^*$ and distinguish the following cases:

Case A, $n = n(x^*) < \infty$. It is easy to check that in this case $R_n\phi(x_\alpha^*) \to R_n\phi(x^*)$. Thus every ω^* limit point of $\{\phi(x_\alpha^*)\}$ is a norm one extension of $R_n\phi(x^*)$ to all of X. By the strict convexity such an extension is unique, and is necessarily equal to $\phi(x^*)$. Thus $\phi(x_\alpha^*) \to^{\omega^*} \phi(x^*)$.

Case B, $\|R_n x^*\| < 1$ for all n. Since $\{\phi(x_\alpha^*)\}$ is bounded, it is enough to show that for each N and all $x \in E_N$, $\langle x, \phi(x_\alpha^*)\rangle \to \langle x, \phi(x^*)\rangle$. Fixing N, $\|R_N x^*\| < 1$, and thus also $\|R_N x_\alpha^*\| < 1$ for all $\alpha > \alpha_0$ for some α_0. But then $\phi(x_\alpha^*)$ is an extension of $R_N x_\alpha^*$ (and of course $\phi(x^*) = x^*$ is an extension of $R_N x^*$). Since $R_N x_\alpha^* \to R_N x^*$ we see that for each $x \in E_N$,

$$\langle x, \phi(x_\alpha^*)\rangle = \langle x, R_N x_\alpha^*\rangle \to \langle x, R_N x^*\rangle = \langle x, \phi(x^*)\rangle.$$

COROLLARY 3.2. *Let* $1 < p < \infty$; *then* L_p *admits a uniformly simultaneously continuous retraction with norm-modulus of continuity* $\omega_p(t)$ *satisfying*

$$\begin{aligned} \omega_p(t) &\leqslant c_p t^{1/p}, \qquad p \geqslant 2, \\ &\leqslant c_p t^{1/2}, \qquad p \leqslant 2. \end{aligned}$$

Proof. This follows from the known estimates for the modulus of convexity of L_p (see, e.g., [4, p. 128]).

Remark. We shall see in the next section that these results are best possible (up to the constants c_p involved). Since l_p and L_p have the same modulus of convexity, we obtain the same results for l_p as well. But as we have seen in Section 2, this is not the best result for l_p when $p < 2$.

The proof of the next theorem follows the same lines as that of Theorem 3.1, and will not be given. Recall that a Banach space Y is locally uniformly convex if for all $y_n, y \in B(Y)$, $\|y_n + y\| \to 2$ implies that $\|y_n - y\| \to 0$.

THEOREM 3.3. *Let* X *be a separable Banach space with a locally uniformly convex dual. Then* X^* *admits a simultaneously continuous retraction.*

Remark. If X is separable and X^* is locally uniformly convex, then X^* is also separable (see, e.g., [3, pp. 31–32]). On the other hand, when X^* is separable, X can be renormed so that X^* under the new norm is locally uniformly convex [3, p. 118]. Combining this with Theorem 3.3 we see that every space X with a separable dual can be renormed so that X^* (under the new norm) will admit a simultaneously continuous retraction.

4. RETRACTIONS IN $C(K)^*$

In this section we show that when K is a compact metric space, $C(K)^*$ admits a norm-Lipschitz ω^* continuous retraction. The main step in the proof is for the special case where K is the Cantor set. The general case follows easily. The retraction constructed here is a modification of the one constructed in [1]. We did not check if the retraction constructed there satisfies a norm-Lipschitz condition.

Let $\Delta = \Delta_{0,1}$ be the Cantor set, and for $n = 1, 2,\dots$ let $\{\Delta_{n,i}\}_{i=1,\dots,2^n}$ be the natural partition of Δ into 2^n disjoint open and closed subsets with $\Delta_{n,i} = \Delta_{n+1,2i-1} \cup \Delta_{n+1,2i}$. By $\{f_{n,i}\}$ we denote the Haar functions

$$\begin{aligned} f_{n,i}(t) &= 1, && t \in \Delta_{n+1,2i-1}, \\ &= -1, && t \in \Delta_{n+1,2i}, \\ &= 0, && t \notin \Delta_{n,i}. \end{aligned}$$

We also put $f_{-1,1} \equiv 1$.

Each $f_{n,i}$ is considered as an element of $C(\Delta)^*$ by identifying it with the measure $f_{n,i}\,d\lambda$, where λ is the normalized Haar measure on Δ.

We order the pairs (n, i) lexicographically, i.e., $(n, i) < (m, j)$ iff $n < m$ or $n = m$ and $i < j$. This is a linear ordering which we "code" by identifying the pair (n, i) with the integer $k = 2^n + i$ $(n = 0, 1, \dots, i = 1, 2, \dots, 2^n)$. We also code $(-1, 1)$ as 1. Using this convention, we shall use the natural numbers as indices of the Haar functions and the intervals of the partition of the Cantor set

$$\begin{aligned} f_k(t) &= 1, && t \in \Delta_{2k-1}, \\ &= -1, && t \in \Delta_{2k}, \\ &= 0, && t \notin \Delta_k. \end{aligned}$$

In this order $\{f_n\}$ is an ω^*-basis for $C(\Delta)^*$: each $\mu \in C(\Delta)^*$ has a unique representation $\mu = \sum \alpha_j f_j$, where the series is ω^* convergent. This is a monotone basis, i.e., the associated projections $P_n(\mu) = \sum_{j \leqslant n} \alpha_j f_j$ satisfy $\|P_n\| = 1$.

We shall use the metric $d(\mu, \nu) = \sum 2^{-n} |(\mu - \nu)| (\Delta_n)$ which induces the ω^* topology on $B(C(\Delta)^*)$.

THEOREM 4.1. *Let K be a compact metric space. Then $C(K)^*$ admits an ω^* continuous retraction ϕ with $\omega_\phi(t) \leqslant 2t$, i.e., $\|\phi(\mu) - \phi(\nu)\| \leqslant 2\|\mu - \nu\|$ for all $\mu, \nu \in C(K)^*$.*

Proof. By Mulitin's lemma [4], $C(K)$ is isometric to a norm one complemented subspace of $C(\Delta)$. Thus Lemma 1.1 implies that it is enough to prove the Theorem for $K = \Delta$.

The retraction ϕ for $C(\Delta)^*$ will be constructed as a limit of ω^* continuous functions ϕ_n: $C(\Delta)^* \to B(C(\Delta)^*)$ which we now define inductively.

Fix $\mu = \sum \alpha_j f_j$. Then put

$$\begin{aligned} \phi_1(\mu) &= \alpha_1 f_1, && |\alpha_1| \leqslant 1, \\ &= \alpha_1 f_1 / |\alpha_1|, && |\alpha_1| > 1. \end{aligned}$$

Having defined $\phi_{n-1}(\mu)$ so that $\|\phi_{n-1}(\mu)\| \leqslant 1$, we distinguish two cases:

If $\|\phi_{n-1}(\mu) + \alpha_n f_n\| \leqslant 1$, put $\phi_n(\mu) = \phi_{n-1}(\mu) + \alpha_n f_n$.

If $\|\phi_{n-1}(\mu) + \alpha_n f_n\| > 1$, put $\phi_n(\mu) = \phi_{n-1}(\mu) + t\alpha_n f_n$, where $t = \inf\{0 < \tau < 1 : \|\phi_{n-1}(\mu) + \tau\alpha_n f_n\| > 1\}$.

It will be important to understand the definition of ϕ_n more directly. Let A be the constant value of $\phi_{n-1}(\mu)$ on Δ_n. If $\|\phi_{n-1}(\mu)\| = 1$, the second case happens iff $|\alpha_n| > |A|$, and in this case $t = |A\alpha_n^{-1}|$. Assuming, for example, that A and α_n have the same sign, we will have that $\phi_n(\mu) \equiv 2A$ on Δ_{2n-1} and $\phi_n(\mu) \equiv 0$ on Δ_{2n}. (When A and α_n have opposite signs, the roles of Δ_{2n} and Δ_{2n-1} interchange).

It is easy to check that each ϕ_n is ω^* continuous. Also for each fixed μ the sequence $\phi_n(\mu)$ is ω^* convergent to a limit which we denote by $\phi(\mu)$. Since $\phi(\mu)$ and $\phi_n(\mu)$ have the same first n coordinates, $d(\phi_n(\mu), \phi(\mu)) \leqslant 2^{-n}$. Thus the sequence $\{\phi_n\}$ converges uniformly to ϕ with respect to d, and ϕ is ω^* continuous. It is clearly a retraction on $B(C(\Delta)^*)$.

We introduce the following notation: If $\mu = \sum \alpha_j f_j$, we put $\psi_n(\mu) = \phi_n(\mu) + (I - P_n)\mu = \phi_n(\mu) + \sum_{j>n} \alpha_j f_j$. Thus $\psi_n(\mu) \neq \psi_{n-1}(\mu)$ iff the nth coordinate of $\phi_n(\mu)$ is different from α_n. It will also be convenient to denote $P_n\mu$ by μ_n, i.e., $\mu_n = \sum_{j \leqslant n} \alpha_j f_j$.

Before we estimate the norm-Lipschitz constant of ϕ we make two observations. First, using the ω^* continuity of ϕ, it is enough to prove that $\|\phi(\mu) - \phi(\nu)\| \leqslant 2\|\mu - \nu\|$ whenever there is an N so that $\mu = \sum_{j \leqslant N} \alpha_j f_j$ and $\nu = \sum_{j \leqslant N} \beta_j f_j$. Indeed once this is proved we have for any μ, $\nu \in C(\Delta)^*$ that $\mu_N \to^{\omega^*} \mu$ and $\nu_N \to^{\omega^*} \nu$, hence $\|\phi(\mu) - \phi(\nu)\| \leqslant \liminf \|\phi(\mu_N) - \phi(\nu_N)\| \leqslant 2 \liminf \|\mu_N - \nu_N\| = 2\|\mu - \nu\|$.

The second observation is that the Lipschitz constant of a function is determined locally. Thus we shall fix $\mu = \sum_{j \leqslant N} \alpha_j f_j$ and prove that $\|\phi(\mu) - \phi(\nu)\| \leqslant 2\|\mu - \nu\|$ only for measures $\nu = \sum_{j \leqslant N} \beta_j f_j$ which are close enough to μ so that the following three conditions are satisfied:

(a) If j is such that $\|\mu_j\| < 1$ (respectively $\|\mu_j\| > 1$), then also $\|\nu_j\| < 1$ (resp. $\|\nu_j\| > 1$).

(b) The two coefficients α_j and β_j have the same sign for each $j = 1,\ldots, N$.

(c) For each dyadic interval Δ_j, $j \leqslant N$, let A_j be the constant value of $\phi_{j-1}(\mu)$ on Δ_j and B_j the constant value of $\phi_{j-1}(\nu)$ on Δ_j. Then A_j and B_j have the same sign.

Now fix, μ, ν as above. Since the sequence of norms $\|\mu_j\|$ is nondecreasing, we see from (a) that there is a $k \leqslant N$ so that $\|\nu_{k-1}\| < 1$, $\|\nu_k\| \geqslant 1$, and $\|\mu_{k-1}\| \leqslant 1$, $\|\mu_k\| \geqslant 1$.

The proof will be done in two steps: First we prove that $\|\psi_k(\mu) - \psi_k(\nu)\| \leqslant 2\|\mu - \nu\|$ and then that $\|\phi(\mu) - \phi(\nu)\| \leqslant \|\psi_k(\mu) - \psi_k(\nu)\|$.

Step 1. Since $\|\mu_{k-1}\|, \|\nu_{k-1}\| \leqslant 1$, we have $\phi_{k-1}(\mu) = \mu_{k-1}$ and $\phi_{k-1}(\nu) = \nu_{k-1}$. Let A (resp. B) be the constant value of μ_{k-1} (resp. ν_{k-1}) on Δ_k. By (b) A and B have the same sign, so assume both are nonnegative. Similarly, by (c) we can assume that $\alpha_k, \beta_k \geqslant 0$.

We claim that without loss of generality it can be assumed that either $\alpha_k \leqslant A$ or $B_k \leqslant B$. Indeed, if $\alpha_k > A$ and $\beta_k > B$, put $t = \min(\alpha_k - A, \beta_k - B)$. Passing from ν to $\tilde{\mu} = \mu - tf_k$ and from ν to $\tilde{\nu} = \nu - tf_k$, we notice that $\|\tilde{\mu} - \tilde{\nu}\| = \|\mu - \nu\|$ and $\phi_k(\tilde{\mu}) = \phi_k(\mu)$, $\phi_k(\tilde{\nu}) = \phi_k(\nu)$; thus, of course, $\|\psi_k(\tilde{\mu}) - \psi_k(\tilde{\nu})\| = \|\psi_k(\mu) - \psi_k(\nu)\|$. But by replacing μ, ν by $\tilde{\mu}$, $\tilde{\nu}$ the condition will already be satisfied.

So assume that $\beta_k \leqslant B$, in which case $\|\nu_k\| = 1$, $\phi_k(\nu) = \nu_k$, and $\psi_k(\nu) = \nu$. If also $\alpha_k \leqslant A$, then $\psi_k(\mu) = \mu$ and there is nothing to prove, so assume $\alpha_k > A$, and then $\phi_k(\mu) = \mu_{k-1} + \alpha f_k$, where $A \leqslant \alpha \leqslant \alpha_k$.

Direct computation shows that in this case $\|\psi_k(\mu) - \mu\| = \|\mu_k - \phi_k(\mu)\| = (\alpha_k - \alpha)\,|\Delta_k|$ (where $|\Delta_k|$ is the measure of Δ_k, i.e., $|\Delta_k| = 2^{-n}$, where $k = 2^n + i$). On the other hand, another direct computation shows that also $\|\mu_k\| - 1 = (\alpha_k - \alpha)\,|\Delta_k|$. Since $\|\nu_k\| = 1$, we get that $\|\psi_k(\mu) - \mu\| = (\alpha_k - \alpha)\,|\Delta_k| = \|\mu_k\| - 1 = \|\mu_k\| - \|\nu_k\| \leqslant \|\mu_k - \nu_k\| \leqslant \|\mu - \nu\|$. Recalling that $\psi_k(\nu) = \nu$, we finally get that $\|\psi_k(\mu) - \psi_k(\nu)\| = \|\psi_k(\mu) - \nu\| \leqslant \|\psi_k(\mu) - \mu\| + \|\mu - \nu\| \leqslant 2\,\|\mu - \nu\|$.

Step 2. Replacing μ by $\mu + (\phi_k(\mu) - \mu_k)$ and ν by $\nu + (\phi_k(\nu) - \nu_k)$ does not change $\phi(\mu)$ and $\phi(\nu)$ respectively and reduces the second step to proving the following claim:

Claim. Assume $\mu = \sum_1^N \alpha_j f_j$, $\nu = \sum_1^N \beta_j f_j$ satisfy (b) and (c) above, and assume there is a $k \leqslant N$ so that $\|\mu_k\| = \|\nu_k\| = 1$. Then $\|\phi(\mu) - \phi(\nu)\| \leqslant \|\mu - \nu\|$.

We shall prove this claim by (inverse) induction on k. It obviously holds for $k = N$, because in this case $\phi(\mu) = \mu$ and $\phi(\nu) = \nu$ and there is nothing to prove. Thus assume it holds for $k + 1$ and we shall prove it for k.

Let μ, ν be as in the claim, and let A (resp. B) be the constant value of μ_k (resp. ν_k) on Δ_{k+1}. As in Step 1, we can assume that A, B, α_{k+1}, and β_{k+1} are all nonnegative and that $\beta_{k+1} \leqslant B$. Of course if $\alpha_{k+1} \leqslant A$, we shall also have $\|\nu_{k+1}\| = \|\mu_{k+1}\|$ and the Claim follows by the induction hypothesis. So assume $\alpha_{k+1} > A$. In this case $\phi_{k+1}(\mu) \equiv 0$ on $\Delta_{2(k+1)}$, and by the definition of ϕ, this implies that we shall also have that $\phi(\mu) \equiv 0$ on $\Delta_{2(k+1)}$.

On the other hand, $\nu_{k+1} \equiv \phi_{k+1}(\nu) \equiv B - \beta_{k+1} \geqslant 0$ on $\Delta_{2(k+1)}$ and the definition of ϕ implies that the norm of $\phi(\nu)|_{\Delta_{2(k+1)}}$, the restriction of $\phi(\nu)$ to $\Delta_{2(k+1)}$, remains the same as that of $\nu_{k+1}|_{\Delta_{2(k+1)}}$, the restriction of ν_{k+1} to $\Delta_{2(k+1)}$. That is, $\|\phi(\nu)|_{\Delta_{2(k+1)}}\| = (B - \beta_{k+1})\,|\Delta_{2(k+1)}|$.

Let $J_1 = \{j: \Delta_j \subseteq \Delta_{2k+1}\}$; $J_2 = \{j: \Delta_j \subseteq \Delta_{2(k+1)}\}$ and put $\mu_1 = \sum_{j \in J_1} \alpha_j f_j$ and $\mu_2 = \sum_{j \in J_2} \alpha_j f_j$. Define ν_1, ν_2 similarly.

We claim that without loss of generality $\mu_2 = \nu_2 = 0$. Indeed, consider $\tilde{\mu} = \mu - \mu_2$, $\tilde{\nu} = \nu - \nu_2$. Then $\|\tilde{\mu} - \tilde{\nu}\| \leqslant \|\mu - \nu\|$ (because $(\tilde{\mu} - \tilde{\nu})$ is just the conditional expectation of $(\mu - \nu)$ with respect to the field in which the subsets of Δ_{2k+1}, and $\Delta_{2(k+1)}$ respectively are identified to two atoms).

Also $\|\phi(\tilde{\mu}) - \phi(\tilde{\nu})\| = \|\phi(\mu) - \phi(\nu)\|$ because $\tilde{\mu} = \mu$ off $\Delta_{2(k+1)}$, so that also $\phi(\tilde{\mu}) = \phi(\mu)$ there, and similarly $\phi(\tilde{\nu}) = \phi(\nu)$ off $\Delta_{2(k+1)}$. On $\Delta_{2(k+1)}$ we know that $\phi(\mu) \equiv \phi(\tilde{\mu}) \equiv 0$ while $\|\phi(\nu)|_{\Delta_{2(k+1)}}\| = (B - \beta_{k+1})\,|\Delta_{2(k+1)}| = \|\phi(\tilde{\nu})|_{\Delta_{2(k+1)}}\|$. Thus if $\|\phi(\tilde{\mu}) - \phi(\tilde{\nu})\| \leqslant \|\tilde{\mu} - \tilde{\nu}\|$, we shall certainley have also that $\|\phi(\mu) - \phi(\nu)\| \leqslant \|\mu - \nu\|$, and by replacing μ, ν by $\tilde{\mu}$, $\tilde{\nu}$ we can already assume that $\mu_2 = \nu_2 = 0$.

Consider now the measure $\eta = \mu - (\alpha_{k+1} - A) f_{k+1}$. This measure satisfies $\|\eta_{k+1}\| = 1$ and of course $\phi(\mu) = \phi(\eta)$. Thus by the induction hypothesis $\|\phi(\mu) - \phi(\nu)\| = \|\phi(\eta) - \phi(\nu)\| \leqslant \|\eta - \nu\|$ and the proof will be finished once we show that $\|\eta - \nu\| \leqslant \|\mu - \nu\|$.

To this end we compute separately the norms of the restrictions of $\eta - \nu$ to $\Delta_{2(k+1)}$, Δ_{2k+1}, and the complement of Δ_k.

(i) Since $\mu_2 = \nu_2 = 0$, we have that $\mu \equiv A - \alpha_{k+1}$, $\nu \equiv B - \beta_{k+1}$, and $\eta \equiv 0$ on $\Delta_{2(k+1)}$. Since also $B - \beta_{k+1} \geqslant 0$ and $A - \alpha_{k+1} \leqslant 0$, we see that $\|(\eta - \nu)|_{\Delta_{2(k+1)}}\| = (B - \beta_{k+1})\,|\Delta_{2(k+1)}|$ and

$$\begin{aligned} \|(\mu - \nu)|_{\Delta_{2(k+1)}}\| &= |(A - \alpha_{k+1}) - (B - \beta_{k+1})| \cdot |\Delta_{2(k+1)})| \\ &= (\alpha_{k+1} - A)\,|\Delta_{2(k+1)}| + \|(\eta - \nu)|_{\Delta_{2(k+1)}}\|. \end{aligned}$$

(ii) On Δ_{2k+1} we have $\mu = A + \alpha_{k+1} + \mu_1$, $\nu = B + \beta_{k+1} + \nu_1$, and $\eta = 2A + \mu_1$. Thus

$$\begin{aligned} &\|(\eta - \nu)|_{\Delta_{2k+1}}\| \\ &\quad = \|(2A + \mu_1 - (B + \beta_{k+1} + \nu_1))|_{\Delta_{2k+1}}\| \\ &\quad = \|[(A + \alpha_{k+1} + \mu_1) - (B + \beta_{k+1} + \nu_1) + (A - \alpha_{k+1})]|_{\Delta_{2k+1}}\| \\ &\quad \leqslant \|[(A + \alpha_{k+1} + \mu_1) - (B + \beta_{k+1} + \nu_1)]|_{\Delta_{2k+1}}\| + (\alpha_{k+1} - A)\,|\Delta_{2k+1}| \\ &\quad = \|(\mu - \nu)|_{\Delta_{2k+1}}\| + (\alpha_{k+1} - A)\,|\Delta_{2k+1}|. \end{aligned}$$

(iii) On the complement of Δ_k, $\mu = \eta$. So that $\mu - \nu = \eta - \nu$ there.

Summing the three estimates together (and noticing that $|\Delta_{2k+1}| = |\Delta_{2(k+1)}|$), we get that $\|\eta - \nu\| \leqslant \|\mu - \nu\|$.

This proves the Claim and the Theorem.

5. Lower Estimates and Counterexamples

We start this section with some computations of lower estimates. These show that the estimates obtained in earlier sections are best possible, and are used to construct spaces that do not admit (uniformly) simultaneously continuous retractions.

PROPOSITION 5.1. *There is an absolute constant c, so that for every $1 < p < \infty$ and every ω^* continuous retraction ϕ from l_p onto $B(l_p)$, $\omega_\phi(t) \geqslant ct^{1/p}$ for all $t \leqslant 1/p$.*

Proof. Fix $\delta > 0$ and let $x_n = e_1 + \delta e_n$, $y_n = (1-\delta^p)^{1/p} e_1 + \delta e_n$. Direct computation gives that $\|x_n - y_n\| = 1 - (1-\delta^p)^{1/p} < (2/p)\,\delta^p$ for all $0 < \delta < 2^{-1/p}$. On the other hand, $x_n \to^{\omega} e_1$ and thus by the weak continuity of ϕ, $\phi(x_n) \to^{\omega} \phi(e_1) = e_1$. Since $\|\phi(x_n)\| \leqslant 1$ and l_p is uniformly convex, the weak convergence of $\phi(x_n)$ to e_1 implies that $\|\phi(x_n) - e_1\| \to 0$. Since $\|y_n\| = 1$ implies that $\phi(y_n) = y_n$, we see that $\lim_n \|\phi(x_n) - \phi(y_n)\| = \lim_n \|e_1 - y_n\| \geqslant \delta$. Given any $t \leqslant 1/p$, we now choose δ so that $(2/p)\,\delta^p = t$, and the result follows by taking x_n and y_n for large enough n.

PROPOSITION 5.2. *There is an absolute constant c so that for all $1 < p < \infty$ and every ω^* continuous retraction ϕ from L_p onto $B(L_p)$, $\omega_\phi(t) \geqslant ct^{1/p}$ (for $p \geqslant 2$) and $\omega_\phi(t) \geqslant ct^{1/2}$ (for $p \leqslant 2$) for all small enough t.*

Proof. Since L_p contains a norm one complemented subspace isometric to l_p, any lower estimate for the possible norm-modulus of continuity of a retraction in l_p will also hold in L_p. Thus the case $p \geqslant 2$ follows from Proposition 5.1.

Assume now that $p \leqslant 2$. It will be convenient to use the relation between the norm-modulus of continuity of retractions and the possible degree of ω^* continuous approximation of nearest points. Thus we shall prove the Proposition by showing that if ϕ is an ω^* continuous f-approximate nearest point from L_p to $B(L_p)$, then $f(t) \geqslant ct^{1/2}$ for all small enough t.

To this end, fix any $0 < \delta < 1$ and let $g_n = 1 + \delta r_n$, where $r_n(t) = \operatorname{sign} \sin 2^n \pi t$ is the nth Rademacher function on $[0, 1]$. Direct computation gives $\|g_n\|^p = \frac{1}{2}((1+\delta)^p + (1-\delta)^p) = 1 + p(p-1)\,\delta^2 + O(\delta^3)$, i.e., (recall that $p \leqslant 2$), $d(g_n, B(L^p)) = \|g_n\| - 1 \leqslant c\delta^2$. On the other hand, it is well known that $r_n \to^{\omega} 0$, and the weak continuity of ϕ implies that $\phi(g_n) \to^{\omega} \phi(1) = 1$. Since L_p is uniformly convex, we get that $\|\phi(g_n) - 1\| \to 0$, hence $\lim_n \|\phi(g_n) - g_n\| = \lim_n \|1 - g_n\| = \delta$ and the result follows.

Before we present the first example we recall the following notation: Given a sequence of Banach spaces X_n and $1 \leqslant p \leqslant \infty$ we denote by $(\Sigma \oplus X_n)_p$ the space of all sequences $y=(x_1, x_2,...)$ with $x_n \in X_n$ and with norm $\|y\| = (\sum \|x_n\|^p)^{1/p}$ (and $\|y\| = \sup \|x_n\|$ for $p=\infty$). Its dual, for $p<\infty$ is the space $(\sum \oplus X_n^*)_q$ where $p^{-1}+q^{-1}=1$.

EXAMPLE 5.3. Let $p_n = n/n-1$ and $X=(\sum \oplus l_{p_n})_1$. Then X^* does not admit a simultaneously continuous retraction.

Proof. By the choice of p_n, we have $l_{p_n}^* = l_n$ and thus $X^* = (\sum \oplus l_n)_\infty$. Denote by $R_n: X^* \to l_n$ the natural projection given by $R_n(x_1^*, x_2^*,...) = x_n^*$. By $\{e_j^n\}_{j=1}^\infty$ denote the unit vector basis.in l_n.

Assume now that $\phi: X^* \to B(X^*)$ is an ω^* continuous retraction.

Claim. For every n there is an N so that whenever $y^* = (x_1^*, x_2^*,...)$ satisfies $\|x_k^*\| \leqslant 1$ for $k<n$ and $x_k^* = e_1^k + \frac{1}{2}e_{m(k)}^k$ with $m(k) \geqslant N$ for $k \geqslant n$, then $\|R_n\phi(y^*) - e_1^n\| < \frac{1}{4}$.

Indeed, if this were false, we could find an n so that for each N there would be y_N^* of the above form with $\|R_n\phi(y_N^*) - e_1^n\| \geqslant \frac{1}{4}$. By passing to a subsequence we can assume that y_N^* converges ω^* to an element y^*, and clearly $\|y^*\| \leqslant 1$ and $R_n y^* = e_1^n$. But then $\phi(y^*) = y^*$ and $R_n\phi(y_N^*)_{N\to\infty} \to R_n\phi(y^*) = R_n y^* = e_1^n$. Since l_n is uniformly convex, this implies that $\|R_n\phi(y_N^*) - e_1^n\|_{N\to\infty} \to 0$, contradicting the choice of y_N^*.

Let $N(1) < N(2) < \cdots$ be a sequence so that $N(n)$ satisfies the Claim with respect to n, and define $y^* = (x_1^*, x_2^*,...)$ by $x_n^* = (1-(\frac{1}{2})^n)^{1/n}\ e_1^n + \frac{1}{2}e_{N(n)}^n$. We have $\|x_n^*\|_{l_n} = 1$ for all n, thus also $\|y^*\| = 1$.

We now define for $t=1, 2,...$ $y_t^* = (x_1^*(t), x_2^*(t),...) \in X^*$ by

$$\begin{aligned} x_n^*(t) &= x_n^*, && n<t, \\ &= e_1^n + \tfrac{1}{2}e_{N(n)}^n, && n \geqslant t. \end{aligned}$$

Fixing t, y_t^* is of the form in the Claim (recall that for $n \geqslant t$, $N(n) \geqslant N(t)$) and thus $\|R_t\phi(y_t^*) - e_1^t\| < \frac{1}{4}$.

It is clear that $\|y_t^* - y^*\| \to 0$. Indeed $\|y_t^* - y^*\| = \sup_{n\geqslant t}[1-(1-(\frac{1}{2})^n)^{1/n}]_{t\to\infty} \to 0$.

To see that $\|\phi(y_t^*) - \phi(y^*)\| \not\to 0$, notice that since $\phi(y^*) = y^*$, $R_t\phi(y^*) = R_t y^* = x_t^*$, and thus $\|R_t\phi(y^*) - e_1^t\| = \|x_t^* - e_1^t\|_{l_t} \geqslant \|\frac{1}{2}e_{N(t)}^t\| = \frac{1}{2}$. Thus

$$\begin{aligned} \|\phi(y_t^*) - \phi(y^*)\| &\geqslant \|R_t(\phi(y_t^*) - \phi(y^*))\| \\ &\geqslant \|R_t\phi(y^*) - e_1^t\| - \|R_t\phi(y_t^*) - e_1^t\| \geqslant \tfrac{1}{2} - \tfrac{1}{4} = \tfrac{1}{4}. \end{aligned}$$

Our second example is of a space X, isomorphic to l_2 which does not admit a uniformly simultaneously continuous retraction. This space has strictly monotone basis so that by Theorem 2.1 it does admit a simultaneously continuous retraction. We do not know whether l_2 can be renormed so as not to admit a simultaneously continuous retraction.

For $\infty > p > 1$, let X_p be the space l_2 with the norm $\|x\|_p = (|x_1|^p + (\sum_{j=2}^{\infty} |x_j|^2)^{p/2})^{1/p}$. The space X_p is isomorphic to l_2 with isomorphism constant at most $\sqrt{2}$, and the unit vectors form a strictly monotone basis for X_p. It is easy to check (by the same argument as in Proposition 4.1) that if ϕ is an ω^* continuous retraction from X_p onto $B(X_p)$, then $\omega_\phi(t) \geqslant ct^{1/p}$ for all $t \leqslant 1/p$, where c is a universal constant.

EXAMPLE 5.4. The space $X = (\sum_{p=2}^{\infty} \oplus X_p)_2$ is isomorphic to l_2 and does not admit a uniformly simultaneously continuous retraction.

Proof. Since each X_p is $\sqrt{2}$ isomorphic to l_2, so is X.

The space X contains each X_p as a norm one complemented subspace; thus the lower estimates for the possible norm-modulus of continuity of an ω^* continuous retraction on X_p will also hold for X, i.e., if $\phi: X \to B(X)$ is any weakly continuous retraction, then $\omega_\phi(t) \geqslant ct^{1/p}$ for all p and $t \leqslant 1/p$. Taking $t_p = 1/p$ and letting $p \to \infty$ we see that $\omega_\phi(t)$ does not tend to zero as $t \to 0$, i.e., ϕ is not norm-uniformly continuous.

REFERENCES

1. Y. BENYAMINI, Small into isomorphisms between spaces of continuous functions, *Proc. Amer. Math. Soc.* **83** (1981), 479–485.
2. H. H. CORSON AND J. LINDENSTRAUSS, On simultaneously extension of continuous functions, *Bull. Amer. Math. Soc.* **71** (1964), 542–545.
3. J. DIESTEL, "Geometry of Banach Spaces—Selected Topics," Lecture Notes in Mathematics No. 485, Springer-Verlag, Berlin/New York, 1975.
4. J. LINDENSTRAUSS AND L. TZAFRIRI, "Classical Banach Spaces," Lecture Notes in Mathematics No. 338, Springer-Verlag, Berlin/New York, 1973.